Unsplit asymptotic preserving schemes for hyperbolic systems with relaxation

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Context

— Hyperbolic relaxation systems: $\partial_t \mathbf{W} + \partial_x \mathbf{f}(\mathbf{W}) = \frac{1}{\varepsilon}\mathbf{R}(\mathbf{W})$

with $\mathbf{W} = (\mathbf{W}^{(1)}, \mathbf{W}^{(2)})$ and $\mathbf{R}(\mathbf{W}) = \left(\mathbf{0},\ \mathbf{Q}(\mathbf{W}^{(1)}) - \mathbf{W}^{(2)}\right)$

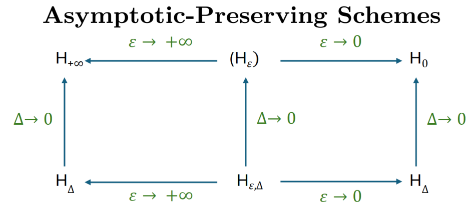
— Asymptotic regime $\varepsilon \to 0$: reduced hyperbolic equilibrium system

$$\partial_t \mathbf{W}^{(1)} + \partial_x \mathbf{f}^{(1)}(\mathbf{W}^{(1)}, \mathbf{Q}(\mathbf{W}^{(1)})) = 0$$

— Entropy structure and subcharacteristic condition

Aims

Asymptotic-Preserving Schemes



Investigate unsplit approaches

— Schemes constructed from the fully coupled relaxation system

— Numerical solver based on the exact integration of the source term

$$\begin{cases} \dfrac{d}{dt}\mathbf{W}^{(2),L,R}(t) = \dfrac{1}{\varepsilon}\left(\mathbf{Q}(\mathbf{W}^{(1),L,R}(t)) - \mathbf{W}^{(2),L,R}(t)\right) \\ \mathbf{W}^{(2),L,R}(0) \text{ is given} \end{cases} \tag{1}$$

→ Future direction: multiple relaxation parameters

Staggered FORCE-type Scheme *[1], [4]*

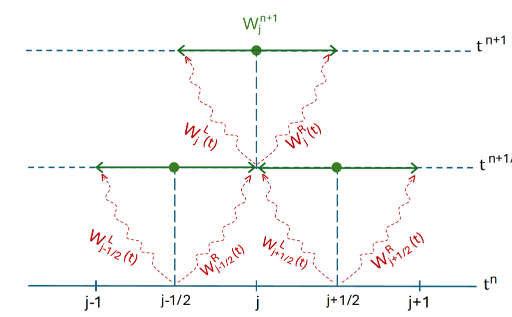
1. **From t^n to $t^{n+1/2} := t^n + \Delta t/2$**
   - *Source term integration*: solve the ODE system (1) with initial data: $\mathbf{W}^L_{j-1/2}(t^n) = \mathbf{W}^n_{j-1}, \mathbf{W}^R_{j-1/2}(t^n) = \mathbf{W}^n_j$
   - *System integration over* $(t^n, t^n + \Delta t/2) \times (x_{j-1}, x_j)$ using the following approximations:

     (a) Fluxes are approximated using the solution of system (1).

     (b) The source term is treated implicitly.

$$\mathbf{W}^{n+1/2}_{j-1/2} = \tfrac{1}{2}(\mathbf{W}^n_{j-1} + \mathbf{W}^n_j) - \tfrac{\Delta t}{2\Delta x}\left[\mathbf{f}(\mathbf{W}^R_{j-1/2}(t^{n+1/2})) - \mathbf{f}(\mathbf{W}^L_{j-1/2}(t^{n+1/2}))\right] + \tfrac{\Delta t}{2\varepsilon}\mathbf{R}(\mathbf{W}^{n+1/2}_{j-1/2})$$

2. **From $t^{n+1/2}$ to t^{n+1}:** repeat steps (a)-(b) over $[x_{j-1/2}, x_{j+1/2}]$ to obtain $\mathbf{W}^{n+1}_j$.



★ The numerical scheme corresponds to an adaptation of the FORCE numerical flux $F^{\text{FORCE}}_{j+\frac{1}{2}} = \dfrac{1}{2}\left(F^{\text{Richtmyer}}_{j+\frac{1}{2}} + F^{\text{Lax-Friedrichs}}_{j+\frac{1}{2}}\right)$, applied to the states $W^{L,R}_{j\pm 1/2}\left(t^{n+\frac{1}{2}}\right)$

Asymptotic preserving property

Let the constant sequence of cell-averaged values $(\mathbf{W}^{(1),n}_j, \mathbf{W}^{(2),n}_j)$ be known at time t^n, for $j \in \mathbb{Z}$. Under the CFL condition $\Delta t \max_{1\le i\le n} \lambda_i \le \Delta x$, the scheme is asymptotic preserving: it remains consistent with the hyperbolic relaxation model for all $\varepsilon > 0$, and converges, as $\varepsilon \to 0$, to the stable and consistent FORCE scheme associated with the limiting hyperbolic equilibrium model.

★ The FORCE scheme, applied to homogeneous hyperbolic system, satisfies a global discrete entropy inequality [4], making the proposed scheme entropy satisfying in limit $\varepsilon \to 0$.

Approximate Riemann Solver (ARS) *[2], [4]*

Build an ARS $\tilde{\mathbf{W}}(x/t; \mathbf{W}_\ell, \mathbf{W}_r)$ to approximate $\mathcal{W}_\mathcal{R}(x, t; \mathbf{W}_\ell, \mathbf{W}_r)$ of the Riemann problem.

1. Solve the ODE system (1) with initial data: $\mathbf{W}^{(2),L}(0) = \mathbf{W}^{(2)}_\ell, \mathbf{W}^{(2),R}(0) = \mathbf{W}^{(2)}_r$

2. Integration over the space-time domain $(-\Delta x/2, \Delta x/2) \times (0, \Delta t)$:

$$\frac{1}{\Delta x}\int_{\frac{-\Delta x}{2}}^{\frac{\Delta x}{2}} \mathcal{W}_\mathcal{R}(x, \Delta t; \mathbf{W}_\ell, \mathbf{W}_r)\mathrm{d}x = \frac{1}{2}(\mathbf{W}_\ell + \mathbf{W}_r) - \frac{1}{\Delta x}\int_0^{\Delta t} \mathbf{f}(\mathcal{W}_\mathcal{R}(\frac{\Delta x}{2}, t; \mathbf{W}_\ell, \mathbf{W}_r)\mathrm{d}t$$
$$+ \frac{1}{\Delta x}\int_0^{\Delta t} \mathbf{f}(\mathcal{W}_\mathcal{R}(-\frac{\Delta x}{2}, t; \mathbf{W}_\ell, \mathbf{W}_r)\mathrm{d}t + \frac{1}{\varepsilon}\frac{1}{\Delta x}\int_0^{\Delta t}\int_{\frac{-\Delta x}{2}}^{\frac{\Delta x}{2}} \mathbf{R}(\mathcal{W}_\mathcal{R}(x, \Delta t; \mathbf{W}_\ell, \mathbf{W}_r))\mathrm{d}x\mathrm{d}t$$

✎ $\int_0^{\Delta t} \mathbf{f}\left(\mathcal{W}_\mathcal{R}\left(\pm\frac{\Delta x}{2}, t; \mathbf{W}_\ell, \mathbf{W}_r\right)\right)dt \simeq \Delta t\, \mathbf{f}\left(\mathbf{W}^{R,L}(\Delta t)\right)$,

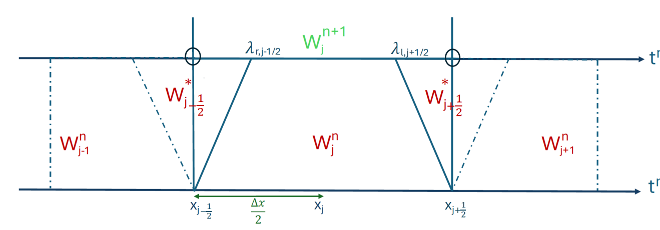
✎ $\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \mathbf{R}\left(\mathcal{W}_\mathcal{R}(x, \Delta t; \mathbf{W}_\ell, \mathbf{W}_r)\right)dx \simeq \{\mathbf{Q}\}_{\ell,r}(\Delta x, \Delta t) - \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \mathcal{W}^{(2)}_\mathcal{R}(x, \Delta t; \mathbf{W}_\ell, \mathbf{W}_r)dx$

Definition of $\{\mathbf{Q}\}_{\ell,r}$

Assume that $\{\mathbf{Q}\}_{\ell,r}$ does not depend on Δt, it holds:

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \tilde{\mathcal{W}}^{(2)}_\mathcal{R}(x, \Delta t; \mathbf{W}_\ell, \mathbf{W}_r)\,dx = \frac{1}{2}(\mathbf{W}^{(2)}_\ell + \mathbf{W}^{(2)}_r)e^{-\Delta t/\varepsilon}$$
$$- \frac{\varepsilon}{\Delta x}(1 - e^{-\Delta t/\varepsilon})\left(\mathbf{f}^{(2)}(\mathbf{W}^L(\Delta t)) - \mathbf{f}^{(2)}(\mathbf{W}^R(\Delta t))\right) + (1 - e^{-\Delta t/\varepsilon})\{\mathbf{Q}\}_{\ell,r}$$

3. Integral consistency relation
4. Assuming that $\{\mathbf{Q}\}_{j-1,j} = \{\mathbf{Q}\}_{j,j+1}$ and replacing $\mathbf{W}^{(2),n+1}_j$ with $\mathbf{Q}(\mathbf{W}^{(1),n+1}_j)$, we obtain $\{\mathbf{Q}\}$



Godunov-type numerical scheme

If $\lambda_{r,j-1/2} = \lambda_{r,j+1/2} = \lambda_R$ and $\lambda_{\ell,j-1/2} = \lambda_{\ell,j+1/2} = \lambda_\ell$:

$$\mathbf{W}^{(1),n+1}_j = \mathbf{W}^{(1),n}_j - \frac{\Delta t}{\Delta x}(\mathbf{F}^{(1),n}_{j+1/2} - \mathbf{F}^{(1),n}_{j-1/2})$$
$$\mathbf{W}^{(2),n+1}_j = \mathbf{W}^{(2),n}_j - \frac{\Delta t}{\Delta x}(\mathbf{F}^{(2),n}_{j+1/2} - \mathbf{F}^{(2),n}_{j-1/2}) - (e^{-\Delta t/\varepsilon} - 1)\left(\mathbf{Q}(\mathbf{W}^{(1),n+1}_j) - \mathbf{W}^{(2),n}_j\right)$$
$$\mathbf{F}^{(1),n}_{j+1/2} = \frac{\lambda_r\lambda_\ell}{\lambda_r - \lambda_\ell}(\mathbf{W}^{(1),n}_{j+1} - \mathbf{W}^{(1),n}_j) - \frac{1}{\lambda_r - \lambda_\ell}\left(\lambda_\ell \mathbf{f}^{(1)}(\mathbf{W}^R_{j+1/2}(\Delta t)) - \lambda_r \mathbf{f}^{(1)}(\mathbf{W}^L_{j+1/2}(\Delta t))\right)$$
$$\mathbf{F}^{(2),n}_{j+1/2} = \frac{e^{\frac{-\Delta t}{\varepsilon}}\lambda_r\lambda_\ell}{\lambda_r - \lambda_\ell}(\mathbf{W}^{(2),n}_{j+1} - \mathbf{W}^{(2),n}_j) + \frac{\varepsilon(e^{-\Delta t/\varepsilon} - 1)}{\Delta t(\lambda_r - \lambda_\ell)}\left(\lambda_\ell \mathbf{f}^{(2)}(\mathbf{W}^R_{j+1/2}(\Delta t)) - \lambda_r \mathbf{f}^{(2)}(\mathbf{W}^L_{j+1/2}(\Delta t))\right)$$

Asymptotic preserving property

Let the constant sequence of cell-averaged values $(\mathbf{W}^{(1),n}_j, \mathbf{W}^{(2),n}_j)$ be known at time t^n, for $j \in \mathbb{Z}$. Under the CFL condition $2\Delta t \max_{j\in\mathbb{Z}}\left(|\lambda_{\ell,j+1/2}|, |\lambda_{r,j+1/2}|\right) \le \Delta x$, the scheme is asymptotic preserving: it remains consistent with hyperbolic relaxation model for all $\varepsilon > 0$, and converges as, $\varepsilon \to 0$,to the stable and consistant HLL scheme [2] for the limit hyperbolic equilibrium model.

★ The scheme satisfies a discrete local entropy inequality up to a numerical error of order $\frac{\Delta t}{\varepsilon}$.

Numerical Results

Compressible two-phase flow model *[3]*

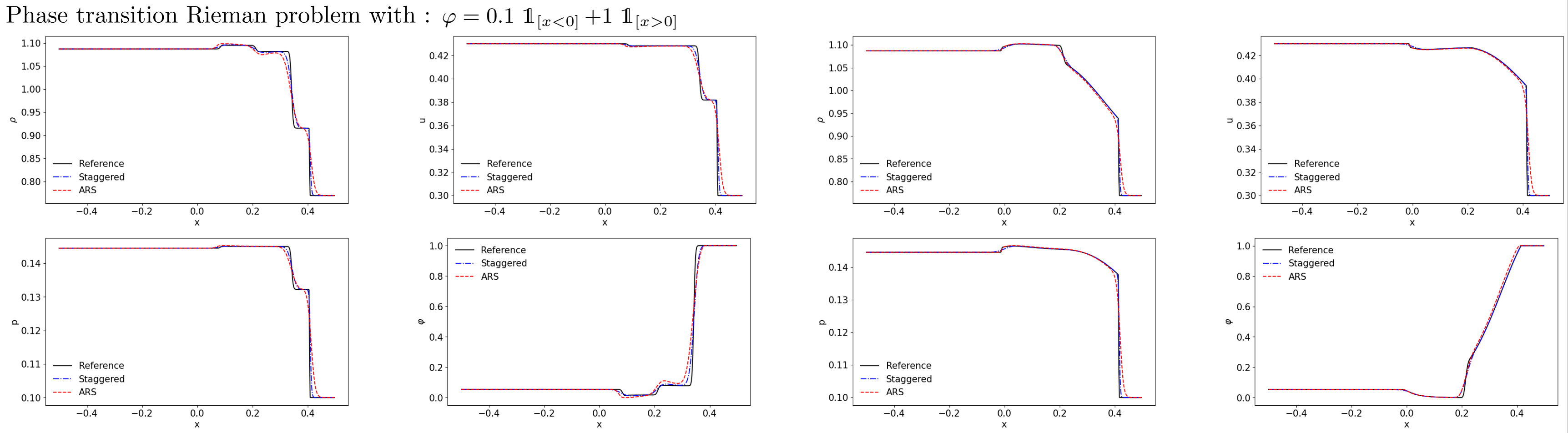
A mixture of two perfect gases characterized by the velocity u, mixture density ρ, internal energy e, total energy E , and mass fraction φ

$$\partial_t \rho + \partial_x(\rho u) = 0$$
$$\partial_t(\rho u) + \partial_x(\rho u^2 + p(\rho, e, \varphi)) = 0$$
$$\partial_t(\rho E) + \partial_x((\rho E + p)u) = 0$$
$$\partial_t(\rho\varphi) + \partial_x(\rho u\varphi) = \frac{\rho}{\varepsilon}(\varphi_{\text{eq}}(\rho) - \varphi)$$

☛ $p(\rho, e, \varphi) = (\gamma(\varphi) - 1)\rho e$, $\gamma(\varphi) = \gamma_1\varphi + \gamma_2(1 - \varphi)$

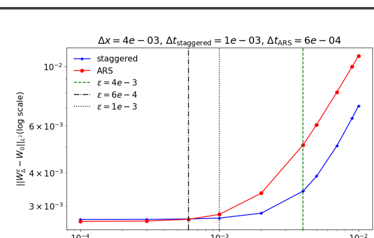
☛ $\varphi_{\text{eq}}(\rho)$: $\rho \in \mathbb{R}^*_+ \to [0, 1]$ — piecewise linear in the mixture zone.

Phase transition Rieman problem with : $\varphi = 0.1\ \mathbb{1}_{[x<0]} + 1\ \mathbb{1}_{[x>0]}$



$\varepsilon = 10^{-15}$ $\varepsilon = 0.1$

The L^2-norm error between the numerical solution of the Jin&Xin relaxation model and the solution of the Burgers equation for small ε.



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